SCALAR QUANTITIES AND VECTOR QUANTITIES IN MECHANICS AND MOTION ANALYSIS

CHAPTER *objectives*

To give students a good basic understanding of vectors and scalars and their application to mechanics.

CHAPTER *outcomes*

After reading this chapter, the student will be able to:

- Distinguish between vectors and scalars.
- Add and subtract vectors by the tip-to-tail method.
- Add and subtract vectors using unit vectors or vector components.
- Resolve a vector into components along orthogonal axes.
- Use the Pythagorean theorem to add two orthogonal vectors.
- Use the cosine rule to add two non-orthogonal vectors.
- Use the scalar or ''dot'' product of two vectors to evaluate work done by a force.
- Use the vector or cross''product of two vectors to evaluate turning moments or torques.

Figure 2.1 Distinguishing between a distance moved (a scalar) and a displacement (a vector).
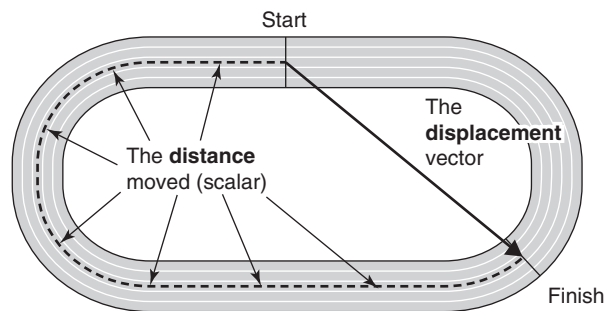
2.1 Addition and Subtraction of Vectors

The study of **mechanics** involves the measurement of physical quantities such as position, mass, velocity, acceleration, force, moment of inertia, time, and angular speed. Some of these quantities are called **scalar** quantities because they can be described completely by single numerical values and are nondirected quantities; mass, time, angular speed, and moment of inertia are all scalar quantities. **Vector** quantities, on the other hand, are completely described by a magnitude and a direction. Displacement, velocity, acceleration, and force are all vector quantities. With vector quantities, the size or magnitude of the quantity must be specified along with the direction.

The difference between a vector and a scalar can be most successfully demonstrated by comparing a displacement and a distance moved. A **displacement** is a vector movement joining the initial position to the final position by a straight line. The **distance**, on the other hand, is a scalar quantity indicating the total distance moved, including any diversions from a straight path. For example, Figure 2.1 shows a runner running three quarters of the way round a 400-m oval track in a time of 40 s. In this example, the displacement, d, is a vector approximately 130 m long pointing in the direction shown, but the distance s moved is 300 m measured all the way along the track. Likewise, we can distinguish between velocity (vector $\mathbf{v}$) and speed: the velocity is displacement divided by time taken and so is a vector in the same direction as $\mathbf{d}$. The speed is the distance traveled divided by the time taken and is a scalar. Therefore, in this example, velocity = 3.25 m/s and speed = 7.5 m/s.

2.1.1 ADDITION OF SCALARS

Scalars can be added by simple addition. Two masses of 50 kg each would make a total of 100 kg. In general, if S_1, S_2, S_3, ..., S_n are scalars to be added together, then the sum of all of them is:

$$S = S_1 + S_2 + S_3 + \ldots\ldots\ldots\ldots\ldots + S_n$$

BOX 2-1	Examples of Scalar Quantities
A mass of 80 kg A moment of inertia of 0.2 kg/m^2	A time of 55 s An angular speed of 10 rad/s (radians per second)

BOX 2-2   ## Examples of Vector Quantities

A velocity of 12 m/s directed due east
A force of 20 N acting vertically upward

An acceleration of 5 ms^2 from left to right
A displacement of 5 m along the line joining A and B

Sometimes this is written in textbooks or scientific publications as:

$$S = \sum_{i=1}^{n} S_i$$

where the symbol Σ stands for summation or "the sum of." The "dummy" index or subscript i is used to show that there are n terms to the summation with i taking on all the whole-number values from 1 to n.

2.1.2 ADDITION OF VECTORS

Vectors add together by simple addition provided that they are both pointing along the same straight line. A displacement of 100 m due south followed by a further displacement of 200 m due south gives a total displacement of 300 m due south. However, vector addition does not proceed so simply if there is an arbitrary angle between the two vectors. We adopt the convention in this book that vectors are represented by bold, italicized capital letters in the text. Two or more such vectors can then be added by the "tip-to-tail" approach. Let's take two vectors, *A* (a displacement of 50 m due north) and *B* (a displacement of 75 m due northeast) and try to add them using the "tip-to-tail" method (Fig. 2.2).

Starting from point P, a person (or any other object) undergoing these two displacements, say *A* first and then *B*, the person or object would move first to point Q and then to point R. The overall displacement is clearly a vector pointing from P (the starting point) to R (the finishing point). Also, the result would still be the same if *B* was executed first and then *A*. We can summarize this as follows:

If the addition of the two vectors is represented by *C*, then:

$$C = A + B$$

BOX 2-3   ## Examples of Scalar Addition

25, 30, and 45 kg add to give 100 kg.

Times of 20, 10, and 15 s add to give 45 s.

Moments of inertia 0.2 and 0.1 kg/m^2 add to give 0.3 kg/m^2.

Angular speeds of 5 and 2.5 rad/s add to give 7.5 rad/s.

CAUTION:

The values of moments of inertia depend on what axis is taken, so that it may not make physical sense to simply add moments of inertia as we have done here. Also, angular speeds, which, although in principle, can be added together simply like this, in practice, other physical laws may be at work, meaning that addition of angular speeds proceeds according to different rules. (See Chapters 6 and 7, which deal with angular kinetics and dynamics.)
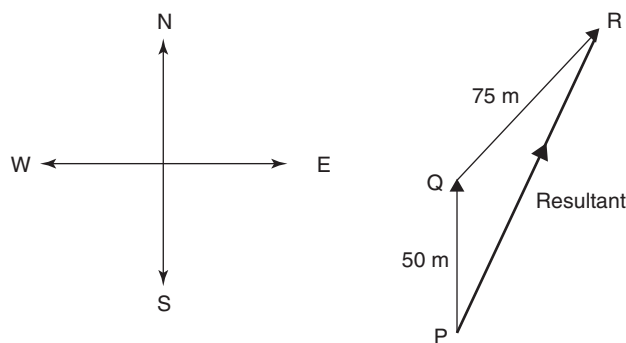
Figure 2.2 Addition of two vector quantities by the tip-to-tail method.

(i.e., resultant displacement PR = displacement PQ + displacement QR) and

A + **B** = **B** + **A** (The order in which the displacements are done is unimportant.)

The result in the above example is 115.88 m at an angle of 27.2° to PQ, but this will not be proved here. This can be shown approximately by a graphical method. Take a piece of graph paper with 1-cm squares, preferably also with 1-mm small squares, and draw the vectors starting from a point near the center of the graph paper, which we will call the origin (0,0). It always makes sense to choose the vertical and horizontal scales according to a "simple-to-use" rule. For example, if we choose each centimeter on the graph paper to represent 10 m, it is easy to draw vectors such as 50 m north because it would be 5 cm on the graph paper and pointing straight up. Also, if we wish to be able to measure angles on the graph paper, then it is essential to make the scales on the horizontal (x) and vertical (y) axes the same. If we wish to draw a vector at some other direction (e.g., for a vector northeast), we would draw it at an angle of 45° using a protractor and pointing in the positive x, positive y direction. Figure 2.3 shows the two vectors, **A** and **B**, and their resultant; in this case, it can be measured with a ruler as being 11.6 cm long and with a protractor as being 27° with respect to **A**.

The magnitude of 11.6 cm translates to a displacement of 116 m and is only as accurate as the length measurement in percentage terms. In this case, the result-
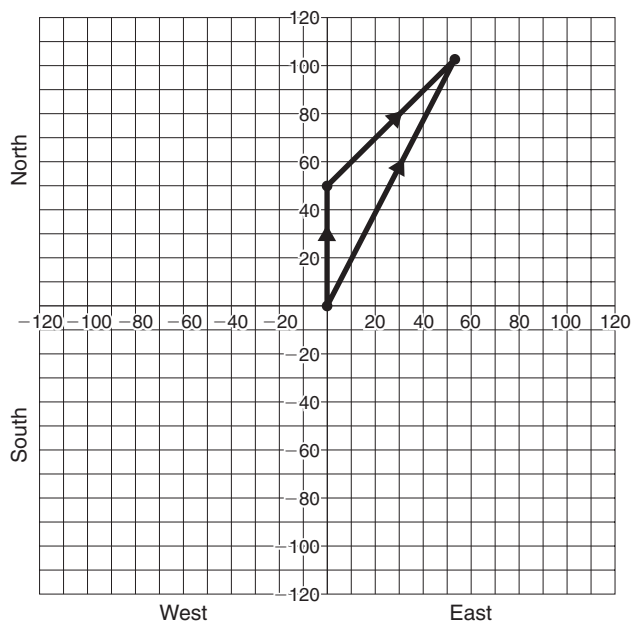

**Figure 2.3** Addition of two displacements **A** and **B** using a graphical method.

Figure 2.4 A football player subject to two forces: his weight acting vertically down from his center of mass and a force exerted by another player on his shirt acting down at an angle of 25° below the horizontal.

ant can only be measured with a ruler to ±0.1 cm, so the displacement can only be quoted to ±1 m, or about ±1%.

It is not always clear how two or more vectors can be added using the tip-to-tail method when the vectors themselves do not appear, at first glance, to be in a tip-to-tail position. Consider the example of the football player shown in Figure 2.4 subject to some shirt tugging. In this unbalanced position, he is likely to fall down and injure himself because of the combined effect of his weight and the force exerted by the other player tugging his shirt. In this case, one or both forces can be translated in the plane of the figure so that they meet tip to tail. The result of this is shown in Figure 2.5 together with the resultant of the vectors *W* and *F*.

Figure 2.5 The two vectors *W* and *F* shown translated, without rotating either of them, and then added to form the resultant using the tip-to-tail method.
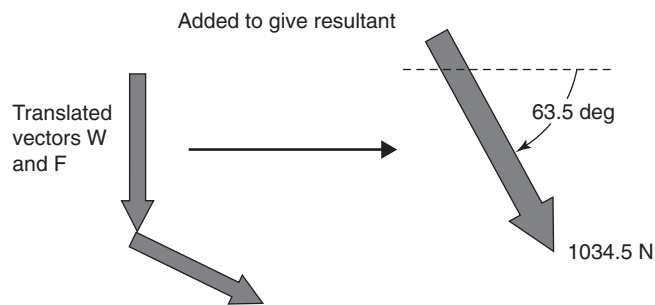
Here are some arbitrary vectors *A* and *B* and their resultant *C*:
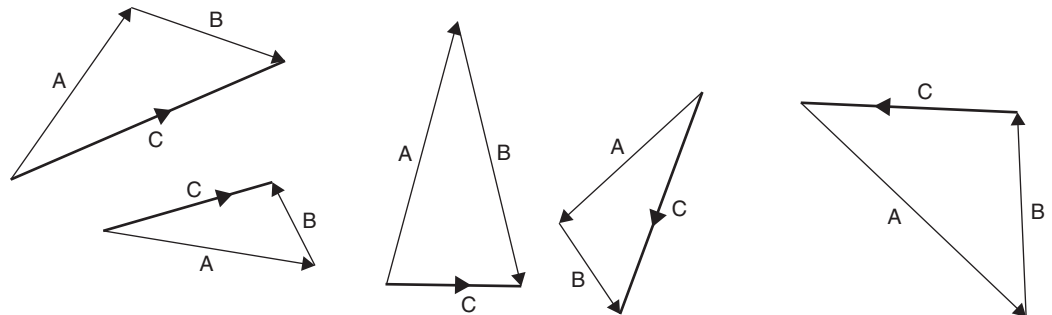
It also follows that any number of vectors *A, B, C, D, ... ,M, N* can be added by placing them tip to tail and producing a polygon of vectors (Fig. 2.6), the resultant *R* being found by completing the polygon to make a closed figure.

The order in which the vectors are added is not important. Again, the vectors can be translated in the plane of the figure to get them to be in the tip-to-tail position.
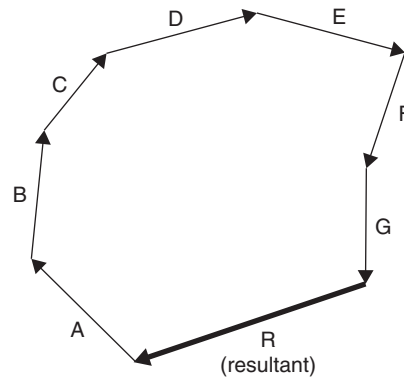
Figure 2.6 Polygon of vectors.

Subtraction of vectors, as in *A* − *B*, can be achieved by adding *A* and (*B* multiplied by scalar −1). This has the effect of changing the direction of the *B* vector by 180° (Fig. 2.7).

■ **EXAMPLE 2.1**

A man throws a baseball with a velocity of 20 m/s at an angle of 30° with respect to the horizontal. Then he does the same throw, but this time he is running with a velocity of 5 m/s in the same direction as the throw. What is the resultant velocity of the ball?
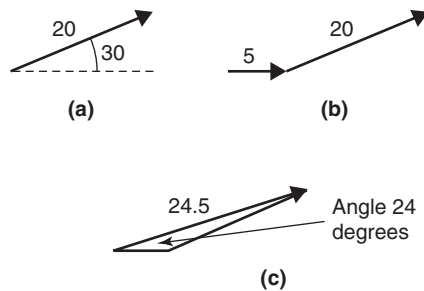
(a) shows the velocity achieved in a normal throw
(b) shows the velocities caused by running and throwing respectively
(c) shows the formation of the resultant by the tip-to-tail method

The resultant velocity with which the ball can be thrown is 24.5 m/s at an angle of 24° with respect to the horizontal, as calculated using the tip-to-tail method on graph paper. (Accurate values for these answers are 24.46 m/s and an angle 24.13°; see section 2.1.4 for the method.)

2.1.3 ADDING TWO VECTORS AT RIGHT ANGLES

When two vectors act at right angles, the resultant can be easily found by adding the vectors using any of the techniques discussed so far. The result can also be
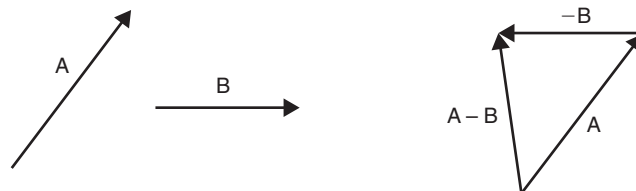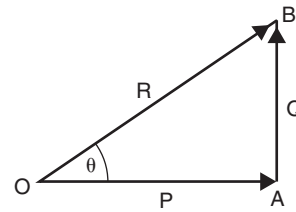

Figure 2.7 Subtraction of two vectors, **A** − **B**.

Figure 2.8 Addition of two vectors, **P** and **Q**, which are perpendicular to each other.

accurately found by using the **Pythagorean theorem** to find the diagonal in the rectangle formed.

Consider a vector **P** represented by OA and a vector **Q** at right angles to **P** represented by AB (Fig. 2.8). The resultant **R** is represented by OB on Figure 2.8. The Pythagorean theorem gives this length as:

$$(OB)^2 = (OA)^2 + (AB)^2$$

or

$$R^2 = P^2 + Q^2$$
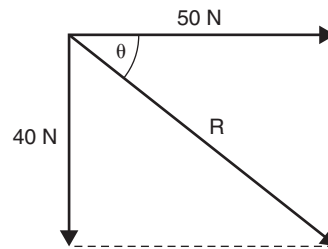
giving

$$R = \sqrt{P^2 + Q^2}$$

.

If the angle θ is required, it is given by:

$$\tan \theta = OA/AB = P/Q$$
$$\theta = \tan^{-1}(P/Q)$$

■ **EXAMPLE 2.2**

An ice skater is propelling herself along a level horizontal ice field with a posterior force of 50 N and a lateral force of 40 N. What is the resultant force of propulsion?

The resultant **R** can be evaluated in terms of magnitude and direction:

Magnitude $R = \sqrt{(50)^2 + (40)^2} = \sqrt{2500 + 1600} = \sqrt{4100} = 64.03N$

Direction $\tan\theta = 40/50 = 0.8$

Therefore, $\theta = \tan^{-1}(0.8) = 38.66°$ or 0.6747 radians

Appendix 1 on Units and Mathematics gives further information on the Pythagorean theorem and trigonometry.

CASE *Study 2.1*   **Magnitude and Direction of Force Vectors During Walking**

An AMTI force plate is used to measure the ground reaction force (GRF) of someone walking across the plate. For this particular force plate, the *z* axis is vertically up, and the *x* and *y* axes

are in the horizontal plane. The following diagram shows the force plate and its relationship with the rectangular coordinate axes. The person walks across the plate in the positive *x* direction.

z direction

x direction
(direction
of walking)

Force plate FP1
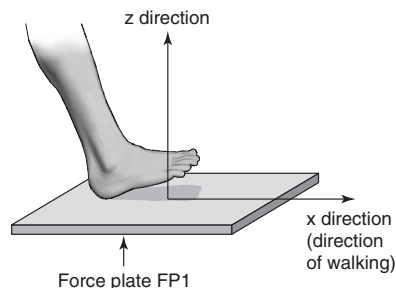
The plate produces voltage outputs corresponding to the *x*, *y*, and *z* components of the GRF as a function of time. The graph shows the components of primary interest, namely the *x* and *z* components of the GRF, using a data collection frequency of 600 Hz for all force components.
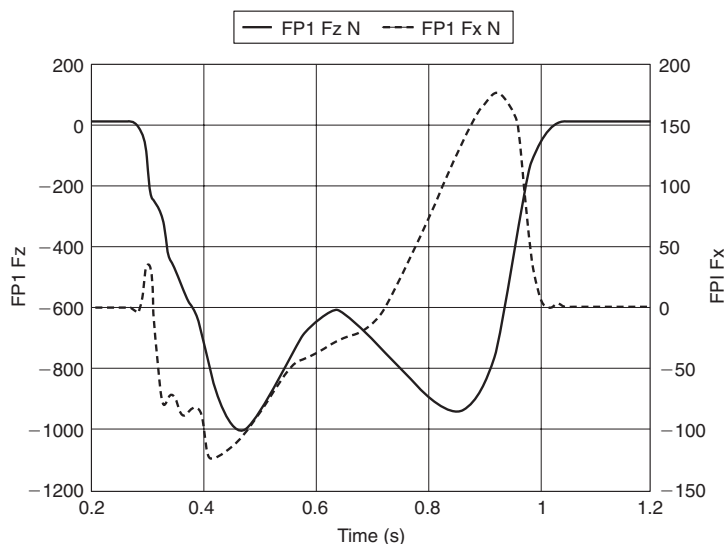
The graph shows the force components F_x and F_x of a normal walking gait. The vertical component F_z is shown as a solid line and has two minima at approximately 0.47 s and 0.86 s. The horizontal component F_x is shown as a dashed line and commences with a mainly negative section before changing to a positive value at approximately 0.72 s. The significance of the positive and negative signs of the force components in this example is clarified by considering the forces acting on the person's foot during the push-off phase.
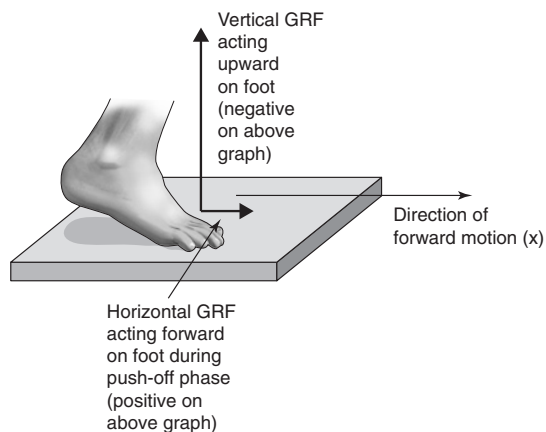

Vertical GRF
acting
upward
on foot
(negative
on above
graph)

Direction of
forward motion (x)

Horizontal GRF
acting forward
on foot during
push-off phase
(positive on
above graph)

Table 2.1	z- and x-force components at five selected times		
Time t (s)	**z-Component of Force (N)**	**x-Component of Force (N)**	**Comment**
0.3583	−501.585	−87.3915	50% of F_z min
0.47	−990.525	−102.3729	Local minimum in F_z
0.64	−605.555	−26.0391	Local minimum in F_z
0.86	−937.135	134.1192	Local minimum in F_z
0.9533	−469.27	159.8016	50% of F_z min

The picture shows the foot in contact with the force plate during the push-off phase showing the x and z force components acting on the foot. Table 2.1 shows the values of F_x and F_z at five selected times.

Using these force components, we can calculate the angle θ and the magnitude of the force **F** at these times. The angle θ at time t = 0.3583 s can be calculated from:

$$\tan \theta = \frac{87.3915}{501.585} = 0.1742$$
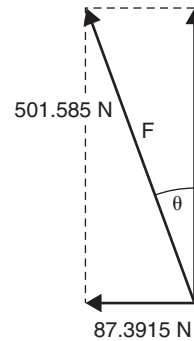
$$\theta = \tan^{-1}(0.1742) = 9.88°$$

Let us call this a positive angle, although such a distinction is arbitrary, with the benefit that if the x component becomes positive, the angle will become negative. The magnitude of the GRF acting on the foot is:

$$F = \sqrt{(501.585)^2 + (87.3915)^2} = 509.14N$$

Likewise, the angles and magnitudes of the force acting on the foot can be calculated at the other times, so giving a table of force magnitudes and angles (Table 2.2). The vector diagram shows the force vector F and its angle of lean roughly to scale.
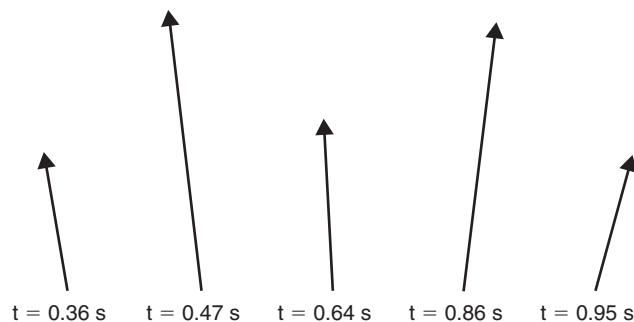

t = 0.36 s   t = 0.47 s   t = 0.64 s   t = 0.86 s   t = 0.95 s

Table 2.2	Magnitude and angle of the resultant force vector at five selected times

Time (s)	Magnitude of GRF (N)	Angle θ(deg)
0.3583	509.14	9.88
0.47	995.80	5.90
0.64	606.11	2.46
0.86	946.68	−8.14
0.9533	495.73	−18.8

GRF = ground reaction force.

2.1.4 PARALLELOGRAM RULE

If OA and AD (Fig. 2.9A) represent two vectors **P** and **Q** making an angle φ with one another, then the resultant **R** can be calculated using the cosine formula:

$$OD^2 = OA^2 + AD^2 - 2OA.AD \cos(180 - \phi)$$

or, using R, P, and Q to stand for the magnitudes OD, OA, and AD, respectively:

$$R^2 = P^2 + Q^2 - 2P.Q \cos(180 - \phi)$$
$$\text{or } R^2 = P^2 + Q^2 + 2P.Q \cos\phi$$

The angle θ between *R* and *P* is given by

$$\tan \theta = \frac{DE}{OE} = \frac{Q \sin\phi}{P + Q \cos\phi}$$

The angle φ between *P* and *Q* appears as an exterior angle in the triangle. For this reason and another reason discussed later, it is often considered best to draw OB = AD to represent *Q* (Fig. 2.9B) so that a completed parallelogram is formed, of which the diagonal through O represents *R*. Thus, if OA and OB represent the size and direction of two vectors **P** and **Q**, their resultant **R** is represented in size and direction by the diagonal OD of the parallelogram OADB.

When dealing with the **cosine rule**, it is essential to label the angles and lengths of the sides of the triangle with a logical system of symbols and to be able to link them with the appropriate formulae. To give an example, let us consider a sprinter on the starting blocks before the starting pistol has been fired. The forces acting on the sprinter are his weight acting vertically down, the reaction force at the blocks,

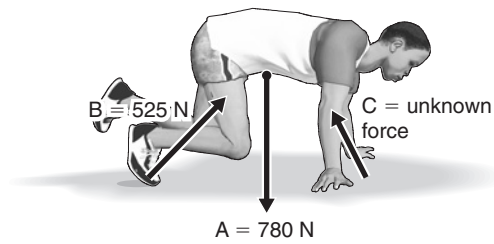**Figure 2.9** (**A**) and (**B**)
The parallelogram rule.
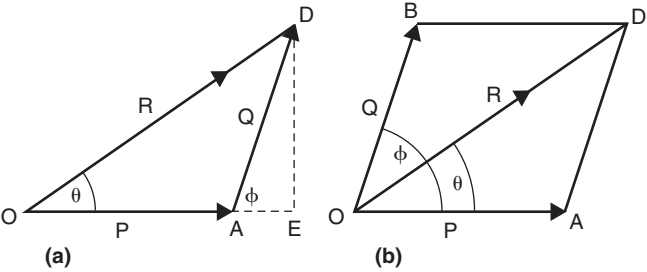
Figure 2.10   The sprinter in the starting blocks experiences three forces: the weight, **A**, acting at the center of mass; the reaction force, **B**, at the blocks acting on his body of magnitude 525 N and at an angle of 70° with respect to the horizontal; and **C**, an unknown force acting at an unknown angle and with an unknown magnitude. The mass of the sprinter is 78 kg, and the magnitude of **A** is, therefore, approximately 780 N. The blocks are instrumented with strain gauges and give an accurate measure of the force **B**.

and the reaction force that the ground exerts on his hands. This example is shown in Figure 2.10, in which *A* represents the weight of the sprinter, *B* represents the reaction force exerted by the blocks on the feet (sum of the left and right feet), and *C* represents the total reaction force exerted by the ground on the hands.

If the sprinter is not moving and not accelerating, the three forces, *A*, *B*, and *C*, will be in equilibrium (see Chapter 3) and will form a closed triangle of forces; there is 0 overall resultant. This fact together with the cosine rule can enable us to find the unknown force *C*.

The forces *A*, *B*, and *C* can be shown on a vector diagram (Fig. 2.11A). Figure 2.11B shows the preferred general labeling of any triangle for application of the cosine rule. Based on Figure 2.11B, the formulas for the cosine rule can be written as:

$$A^2 = B^2 + C^2 - 2BC\cos\alpha$$
$$B^2 = A^2 + C^2 - 2AC\cos\beta$$
$$C^2 = A^2 + B^2 - 2AB\cos\gamma$$
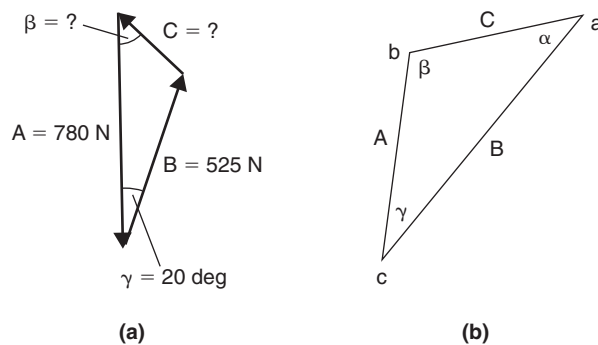

(a)   **(b)**

Figure 2.11   (A) The forces **A**, **B**, and **C** exerted on the body of the sprinter with a resultant of 0. The force **C** is unknown and can be calculated from the cosine rule. This is true provided the sprinter is stationary and not accelerating. (B) The general labeling of a triangle before application of the cosine rule. The lengths of the sides of the triangle are allocated symbols A, B, and C on an arbitrary basis. The angle opposite the A side is labeled α, the angle opposite the B side is labeled β, and the angle opposite the C side is labeled γ. This should be adhered to carefully so that the cosine rule formulae can be applied correctly. If the angles α and β are interchanged, for example, errors may occur in calculation of lengths and angles. If desired, the three apexes can be labeled *a*, *b*, and *c* with apex *a* opposite side A and so on.

It is recommended that these rules are remembered and applied in this form to avoid confusion and to provide a logical basis for calculations. This rule is also summarized in Appendix 1.

In Figure 2.11A, the magnitude of C is unknown. This can be found by applying the cosine rule:

$$C^2 = A^2 + B^2 - 2AB\cos\gamma$$
$$C^2 = (780)^2 + (525)^2 - 2(780)(525)\cos(20)$$
$$C^2 = 608400 + 275625 - 769608.3 = 114416.7$$
$$C = 338.25 \text{ N}$$

In some textbooks, this is sometimes written as $|\underline{C}|$ the magnitude of the vector $\underline{C}$. In this book, the vector is shown in bold italics C and the magnitude of the vector is shown as just C or $|C|$.

To find the angle at which C acts at the hands, the cosine rule can be applied again, or a variety of other methods can be used. Applying the cosine rule:

$$\cos\beta = \frac{-B^2 + A^2 + C^2}{2AC} = \frac{-275625 + 608400 + 114416.7}{2(780)(338.25)} = \frac{447191.7}{528060}$$
$$= 0.8468$$

$$\beta = 32.1°$$

The reaction force at the hands acting on the sprinter's body is of magnitude 338 N and acts at an angle of 57.9° with respect to the horizontal.

■ **EXAMPLE 2.3**

A sailboat moves 1200 m due east followed by 1500 m due northeast. What is the overall displacement?

Solution A: Graphical Method

Draw a vector diagram on graph paper showing the two displacements drawn to scale (e.g., 1 cm to every 100 m.) The initial position of the boat can be taken at the origin of coordinates (0,0) at the bottom lefthand corner of the graph paper.
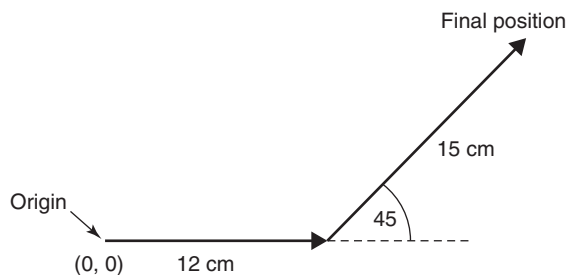
The angle of 45° for the second leg of 1500 m is drawn in accurately using a protractor. The resultant displacement is now represented by the line (not shown) joining the origin to the final position. By measuring with a ruler, this length is 25 cm (or 2500 m ± 10 m), and the angle of the resultant with the horizontal is 25 ± 1°.

Solution B: Cosine Rule
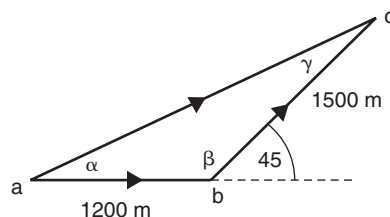
The angle β is $(180 - 45) = 135°$. α and γ are unknown. If the length $ab = C$, $bc = A$ and $ac = B$, then:

$$B^2 = A^2 + C^2 - 2AC \cos \beta \text{ (cosine rule)}$$

so that:

$$B^2 = 1200^2 + 1500^2 - 2(1200)(1500) \cos (135)$$
$$B^2 = 1440000 + 2250000 - 3600000(-0.7071)$$
$$= 1440000 + 2250000 + 2545560$$
$$= 6235560$$
$$B = 2497.1 \text{ m}$$

For a complete solution, one of the angles α or γ (preferably α) would have to be specified in order to describe the direction of the resultant ac. It can be calculated from the cosine rule in this form:

$$A^2 = B^2 + C^2 - 2BC \cos \alpha$$

$$\cos \alpha = \frac{-A^2 + B^2 + C^2}{2BC} = \frac{-2250000 + 6235560 + 1440000}{2(2497.1)(1200)} = \frac{56425560}{5993040} = 0.9053$$

$$\alpha = \cos^{-1}(0.9053) = 25.1°$$

or, alternatively, it can be calculated using

$$\tan \alpha = \frac{(BC)\sin\phi}{(AB) + (BC)\cos\phi} = \frac{1500\sin(45)}{1200 + 1500\cos(45)} = \frac{1060.66}{2260.66} = 0.4692$$

$$\alpha = \tan^{-1}(0.4692) = 25.1°$$

2.2   Resolution of a Vector Into Components

2.2.1 RESOLUTION OF A VECTOR ALONG TWO PERPENDICULAR DIRECTIONS

We can **resolve** any given vector into two others in any two given directions by working the parallelogram rule backward, starting with the diagonal and drawing the adjacent sides in the chosen directions. The most important case is when the two directions are at right angles to each other; the two vectors then obtained are called the **components** of the original vector. When working in three dimensions, a vector can be resolved into three components along three perpendicular axes.

Thus, given a vector R, to find the components in two directions 1 and 2 at right angles to one another, OD is drawn to scale to represent R, the rectangle OADB is constructed where OA represents P, the component in direction 1, and OB represents Q, the component in the direction 2. This procedure is summarized in Fig. 2.12

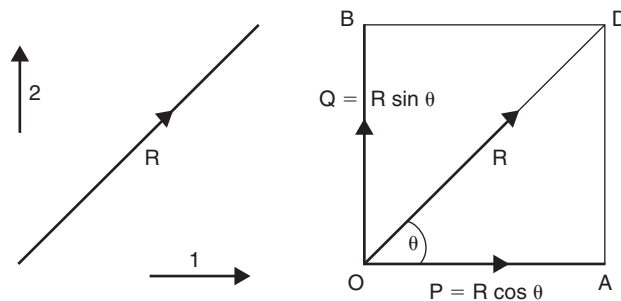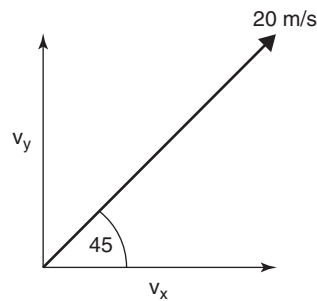**Figure 2.12**   Resolution of **R** in directions 1 and 2 at right angles.

$$\text{As } OA/OD = \cos\theta$$
$$P/R = \cos\theta, \text{ or } P = R\cos\theta$$
$$\text{and as } OB/OD = AD/OD = \sin\theta$$
$$Q/R = \sin\theta, \text{ or } Q = R\sin\theta$$

See also Appendix 1 for a summary of mathematical techniques.

■ **EXAMPLE 2.4**

A javelin is thrown with an initial velocity of 20 m/s at an angle of 45° to the horizontal. Resolve this velocity into vertical and horizontal components.

Solution

The horizontal and vertical components are shown on the diagram as v_x and v_y, respectively.

$$v_x = 20\cos 45 = 20(0.7071) = 14.14 \text{ m/s}$$

and

$$v_y = 20\sin 45 = 20(0.7071) = 14.14 \text{ m/s}.$$

In this example, v_x and v_y are equal. The original velocity of 20 m/s acting at 45° is entirely equivalent to the combined action of its components v_x and v_y and can be replaced by them in all calculations. It should be stressed, however, that you should only resolve once. Also, when components are taken, the original vector should then be forgotten about and ignored because it has been *replaced* by its components.

■ **EXAMPLE 2.5**

An athlete's foot produces a GRF of 410 N horizontally and 520 N vertically. What is the magnitude of the GRF and its angle with respect to the horizontal?

Solution

In this case, the angle θ that the force makes with the horizontal is unknown.
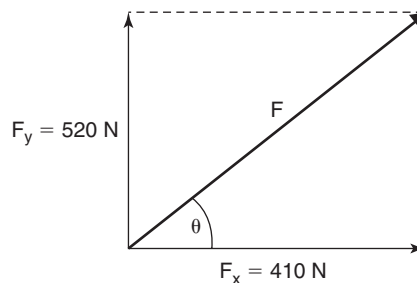
The magnitude of the force F is given by:

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{410^2 + 520^2} \text{ (by the Pythagorean theorem)}$$
$$F = 662.2 \text{ N}$$

The angle at which **F** acts is given by:

$$\tan \theta = \frac{520}{410} = 1.268$$

$$\theta = \tan^{-1}(1.268) = 51.7°$$

2.2.2 UNIT VECTORS IN THREE DIMENSIONS

Vectors can be conveniently expressed in terms of **unit vectors** (meaning vectors whose magnitude is one) pointing along the chosen coordinate axes. With rectangular righthanded x, y, and z axes (Fig. 2.13), the unit vectors are called **i**, **j**, and **k** and point along the positive x, y, and z axes, respectively. This means that any vector **V** can be written as:

$$\mathbf{V} = V_x \mathbf{i} + V_y \mathbf{j} + V_z \mathbf{k}$$

where V_x, V_y and V_z are the components of V along the x, y, and z axes, respectively. This is sometimes abbreviated to just:

$$\mathbf{V} = (V_x, V_y, V_z)$$

This means that any vector can be expressed as V_x units to the right, V_y units up, and V_z units perpendicular to the first two. It also follows that the magnitude of a vector V can be expressed as:

$$V^2 = V_x^2 + V_y^2 + V_z^2 \text{ or } V = (V_x^2 + V_y^2 + V_z^2)^{1/2}$$

To add two vectors **A** and **B**, the resultant can be written as:

$$\mathbf{R} = \mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}$$

2.3   Multiplication of Vectors

2.3.1 MULTIPLICATION OF A VECTOR BY A SCALAR

A vector can be multiplied by a scalar quantity to produce another vector. Thus, if we have a vector **A** and a scalar c, then the product of the two is another vector $c\mathbf{A}$. The properties of $c\mathbf{A}$ are:

1. It has the same direction as the first vector **A** if c is positive and has the opposite direction to **A** if c is negative. If $c = 0$, clearly $c\mathbf{A}$ is 0 as well.
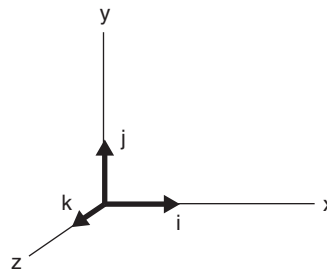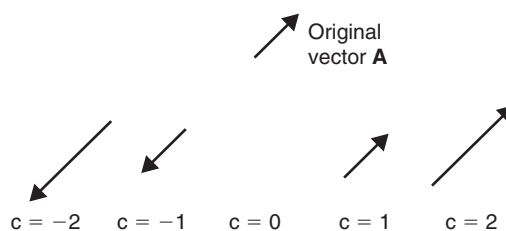
Figure 2.13   Unit vectors pointing along righthanded x, y, and z axes.

Figure 2.14 Illustration of the effect of multiplication by various positive and negative scalars on a vector **A**. The vectors in the lower half of this diagram represent vectors c**A** for c = −2, −1, 0, 1 and 2.


2. Its magnitude is increased or decreased by the magnitude of c. So, if c = 0.5, then the magnitude of the new vector will be one half the magnitude of **A**. If c = −3, then the magnitude of c**A** will be three times that of **A**.

Figure 2.14 shows an arbitrary vector and how it is altered for five different values of the scalar c.

CASE *Study 2.2*   **Higher Velocity with New Golf Club**

A new design of golf club enables professional players to achieve 15% more velocity off the tee on average than with the previous best golf clubs. An average velocity with the new club was 51.42 m/s compared with 44.72 m/s previously. The direction of hitting remains unchanged on average.
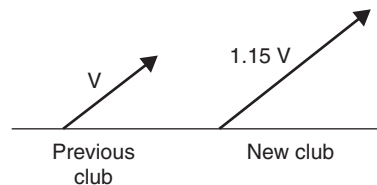
CASE *Study 2.3*   **Velocity of a Baseball at Different Times**

A pitcher pitches a ball in the game of baseball. The velocity components of the ball at three different subsequent times are given in Table 2.3. At $t = 0.675$ s, the ball is in the pitcher's hand and is being accelerated down. At $t = 0.800$ and 0.925 s, the ball is in the air and is moving down toward the ground. The velocity vectors at these times can be written as:

Table 2.3 The x, y, and z components of the ball velocity at three selected times

t (s)	V_x (m/s)	V_y (m/s)	V_z (m/s)
0.675	−2.733	−2.527	−1.863
0.800	−4.008	−4.062	−6.471
0.925	−3.911	−3.961	−7.405

$$v = -2.733\mathbf{i} - 2.527\mathbf{j} - 1.863\mathbf{k}$$
$$v = -4.008\mathbf{i} - 4.062\mathbf{j} - 6.471\mathbf{k}$$
$$v = -3.911\mathbf{i} - 3.961\mathbf{j} - 7.405\mathbf{k}$$

The magnitudes of the resultant velocity vectors are 4.162, 8.628, and 9.264 m/s, respectively.

■ **EXAMPLE 2.6**

A ball experiences two forces: its weight W = 5 N acting vertically down and a drag force directed in a direction opposite to the velocity of the ball. The velocity of the ball is given by $\mathbf{v} = 15\mathbf{i} + 18\mathbf{j} - 20\mathbf{k}$. The drag force is given by:

$$F_d = 0.01 \times (\text{speed})^2$$

Figure out the weight of the ball and the drag force in terms of unit vectors **i**, **j**, and **k**. What is the resultant force on the ball in terms of unit vectors?

Solution

The weight of the ball can be written as:

$$\mathbf{W} = 0\mathbf{i} + 0\mathbf{j} - 5\mathbf{k}$$

The speed of the ball is given by the magnitude of its velocity:

$$v = \sqrt{(15)^2 + (18)^2 + (-20)^2} = \sqrt{225 + 324 + 400} = \sqrt{949} = 30.8 m/s$$

Therefore, the magnitude of the drag force is given by:

$$F_d = 0.01 \times (30.8)^2 = 9.49 N$$

In terms of unit vectors, it can be written as $F_d = (magnitude) \times \hat{v}$, where $\hat{v}$ is a unit vector along the direction of v.
A unit vector along the direction of the velocity would be

$$\hat{v} = \frac{15i + 18j - 20k}{30.8} = 0.487\mathbf{i} + 0.584\mathbf{j} - 0.649\mathbf{k}$$

We can write the drag force as:

$$\mathbf{F_d} = 9.49 \times (0.487\mathbf{i} + 0.584\mathbf{j} - 0.649\mathbf{k})$$
$$\mathbf{F_d} = 4.621\mathbf{i} + 5.542\mathbf{j} - 6.162\mathbf{k}$$

The overall force can be written as:

$$\mathbf{F} = \mathbf{W} + \mathbf{F_d}$$
$$\mathbf{F} = 4.621\mathbf{i} + 5.542\mathbf{j} - 11.162\mathbf{k}$$

2.3.2 SCALAR PRODUCT OF TWO VECTORS

If we have two vectors **A** and **B**, then their **scalar** (or **dot**) **product** is defined as:

$$\mathbf{A.B} = A\,B\cos\theta$$

where A and B are the magnitudes of the vectors and θ is the smallest angle ($< 180°$) between them when their tails touch, as in Figure 2.15. Because A, B, and $\cos\theta$ are all
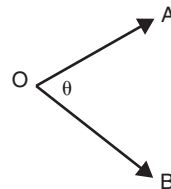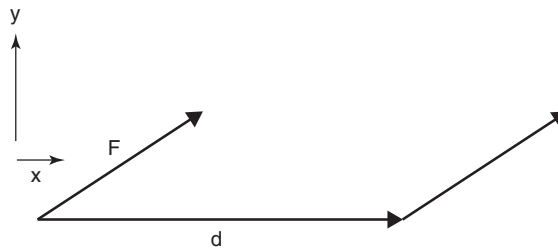

Figure 2.15 Two vectors, **A** and **B**, and the angle between them.

Figure 2.16 Work done by a force **F** acting at an angle θ to the ground is $W = $ **Fd**.

scalars, then so is the scalar product **A.B** (read as **A** "dot" **B**), hence the name scalar product. If the two vectors are expressed in terms of unit vectors, then:

$$\text{Scalar product} = \mathbf{A.B} = A_x B_x + A_y B_y + A_z B_z$$

This is useful in mechanics, particularly when it comes to defining the work done by a force, **F**, that is moved through a displacement, **d** (Fig. 2.16). We would then take the work done, W, as the scalar product of **F** and **d**:

$$W = \mathbf{F.d} = F\,d\cos\theta.$$

If **F** and **d** are expressed using unit vectors, then the work done is:

$$W = \mathbf{F.d} = F_x d_x + F_y d_y + F_z d_z$$

The work done can then be viewed as the magnitude of the force (F) multiplied by the distance moved in the direction of the force (d cos θ). If **F** and **d** are collinear, (θ = 0), then cos θ = 1, and the work done is just force multiplied by distance moved.

▪ **EXAMPLE 2.7**

A man pushing a bobsleigh exerts a force of 550 N acting at an angle of 20° to the horizontal. What is the work done on the bobsleigh over a 20-m distance along the bobsleigh run?
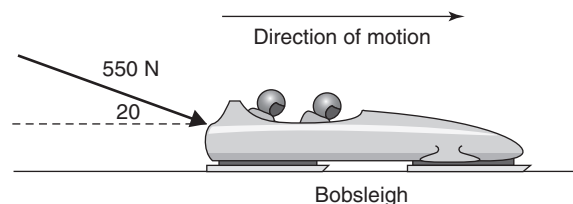
Solution

The work done is **F.d** = F d cos θ where θ is the angle between **F** and **d. d** is a displacement vector parallel to the direction of motion of the bobsleigh. The work done, therefore, is:

$$W = 10336.7 \text{ Joules (or Newton meters)}$$

▪ **EXAMPLE 2.8**

A man pushing a bobsleigh exerts a force of 516.8 N horizontally and a force of 188.1 N vertically downward. The bobsleigh moves horizontally a distance of 20 m down the track. What is the work done on the bobsleigh?

Solution

Using unit vectors **i**, **j**, and **k**, the force exerted on the bobsleigh can be written as:

$$\mathbf{F} = 516.8\mathbf{i} - 188.1\mathbf{j} + 0\mathbf{k}$$

assuming the bobsleigh is moving along the horizontal x axis and the upward vertical direction is the y axis.

The displacement of the bobsleigh can be written as:

$$\boldsymbol{d} = 20\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}.$$

The work done can then be worked out as:

$$W = \boldsymbol{F}.\boldsymbol{d} = F_x d_x + F_y d_y + F_z d_z = 516.8(20) - 188.1(0) + 0(0)$$
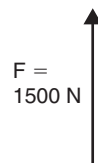$$W = 10336 \text{ J}$$

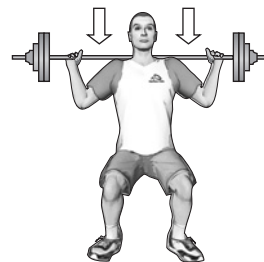| CASE *Study 2.4* | **Downward and Upward Movements in a Squat** |

A weightlifter training for a competition squats 150 kg. On the downward movement, he allows the weight to fall at constant speed slowly to the lowest point by flexing his knees. He then pushes the weight upward at constant speed until his legs are straight. The total displacement is 0.9 m down and then 0.9 m back up. The weightlifter exerts a force **F** upward on the bar, which is roughly constant throughout the downward and upward movement, equal in magnitude roughly to the weight on the bar provided the upward and downward accelerations are small.
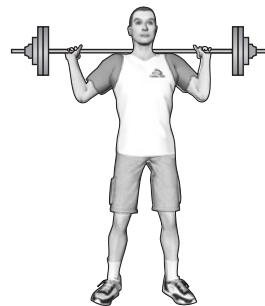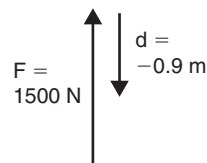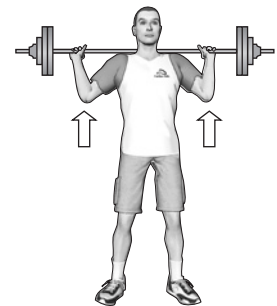

Stationary:
displacement = 0

After downward movement

After movement backup

F =
1500 N

F =
1500 N   d =
              −0.9 m

F =
1500 N   d =
              +0.9 m

Scalar product
F.d =
since d = 0

Scalar product
F.d = F × d × cos (180)
    = (1500) × (0.9) × (−1)
    = −1350 Nm
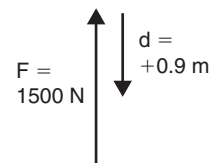
Scalar product
F.d = F × d × cos (0)
    = (1500) × (0.9) × (1)
    = +1350 Nm

This illustrates that, after the downward movement, the force **F** on the bar and the displacement **d** of the bar are acting along the same straight line but in exactly opposite directions. In this case, the angle θ is 180°, the cosine of this angle is −1, and the work done by the weightlifter on the bar is −1350 Nm or Joules.

After the upward movement, the displacement of the bar is now upward, and **F** and **d** are pointing in the same direction. It follows that the angle θ is now 0, the cosine of the angle is 1, and the work done by the weightlifter on the bar is = 1350 Nm or Joules.

In the case of the downward movement, the work done by the weightlifter is negative, and it is natural to ask how this can happen and what it means. It is a difficult concept, but it can be made easier to understand by saying in this case that the bar is doing positive work on the weightlifter. During the upward movement, the reverse is true, and the weightlifter does positive work on the bar. Although the concept may appear a little artificial, it has

important application in eccentric and concentric contraction of muscles. This idea of negative work being done actually occurs whenever the angle θ exceeds 90°. The functional dependence of $\cos\theta$ on θ is shown in the following diagram for values of θ between 0 and 180°, where it can be seen clearly that $\cos\theta$ is negative between 90° and 180°. Incidentally, angles are sometimes expressed in radians; if you need to convert between the two, the conversion factor is:

$$1 \text{ radian} = 57.29°$$

Radian measures are covered more fully in the chapters dealing with rotational motion.
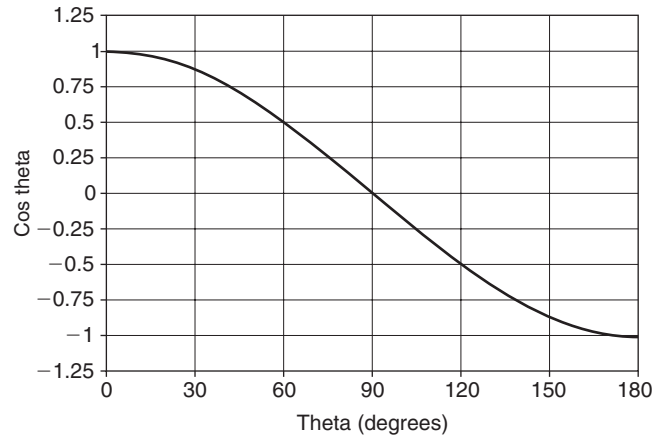
2.3.3 Vector Product of Two Vectors

The **vector** or **cross product** of two vectors *A* and *B* is defined as another vector:

$$C = A \times B$$

whose magnitude is:

$$|C| = |A||B|\sin\theta$$

where θ is the angle ($< 180°$) between *A* and *B*, and whose direction is perpendicular to both *A* and *B* in the sense of the righthand rule (Fig. 2.17).
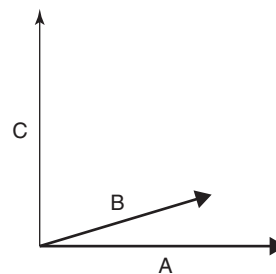
Figure 2.17   The vector **C** = **A** × **B** is perpendicular to the plane containing **A** and **B**; its direction is given by the righthand rule. The thumb of the right hand points along the direction of **C**; the fingers of the right hand curl around from **A** to **B**. Imagine **A** and **B** to be in the plane of the floor, perhaps. **C** would then be pointing vertically up in the air.
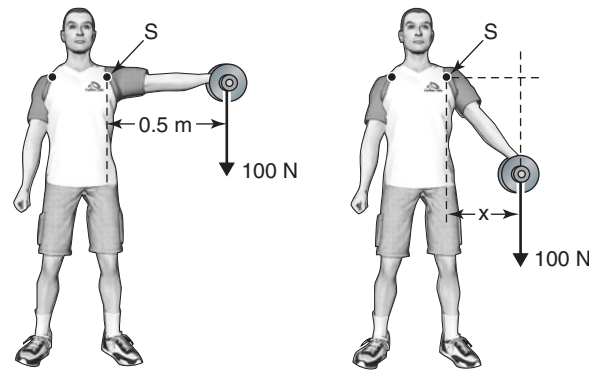
Figure 2.18   Someone holding a 100-N weight at arm's length. The length of the arm is 0.5 m. On the left, the arm is horizontal, and the line of action of the weight is 0.5 m from the shoulder joint. On the left, the arm has been lowered so that it is at 45° to the horizontal. The perpendicular distance between the line of action of the weight and the shoulder joint has now been reduced to x.

If the angle $\theta = 0$, then the crossproduct of two vectors is 0. Hence, $A \times A = 0$, for example. Also, if the angle between two vectors is a right angle, $\theta = 90°$, then $A \times B = AB$, the product of the magnitudes of the two vectors.

This concept is particularly useful when evaluating the moment caused by a given force. Consider someone holding a 10-kg mass at arm's length (Fig. 2.18). We can define a vector r to represent the displacement of the point of application of the weight W relative to the shoulder joint. The **moment** of the force is then given by:

$$M = r \times W$$

This is shown in Figure 2.19 for the two cases shown in Figure 2.18. The results for the vector cross product in the two cases are:

Arm horizontal: $M = rF\sin\theta = 0.5 \times 100 = 50$ Nm

Arm at 45° to horizontal: $M = rF\sin\theta = 0.5 \times 100 \times \sin(45) = 35.35$ Nm

The direction of the moments is clockwise in both Figures 2.18 and 2.19. This would be indicated formally in the vector crossproduct by saying that the vector M is in to the plane of the paper at right angles to both r and W. This is confirmed by using the righthand rule: the fingers curl around in the direction of r moving to the direction of W, meaning that the thumb is pointing in to the plane of the paper (Fig. 2.20).

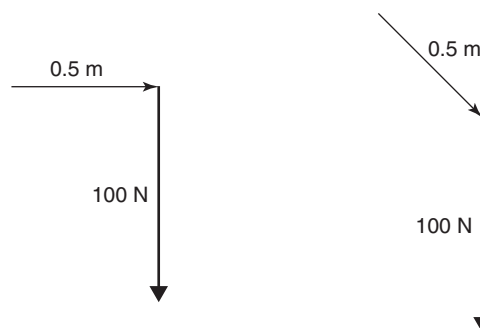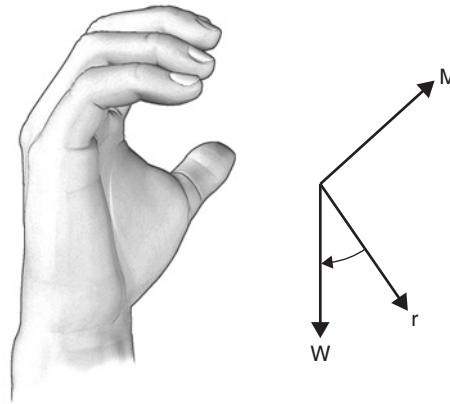**Figure 2.19**   The vector diagrams for the two cases shown in Figure 2.18.

Figure 2.20 The right-hand rule for **r** × **W**, showing the thumb pointing in to the paper.

It is also possible to express the vector product of two vectors using unit vectors. Again suppose that:

$$A = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}$$

and

$$B = B_x\,\mathbf{i} + B_y\,\mathbf{j} + B_z\,\mathbf{k}$$

then the cross- or vector product can be written as:

$$A \times B = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = i(A_y\,B_z - A_z\,B_y) + j(A_z\,B_x - A_x\,B_z) + k(A_x\,B_y - A_y\,B_x)$$

The object:

$$\begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

is known as a **determinant** and is useful in remembering the sequence of calculations required to evaluate the vector crossproduct. Think of it as a sequence of cross-multiplication and addition/subtraction operations (Fig. 2.21). For more information on determinants and the subject of vector analysis, see Spiegel (1959).

Figure 2.21 Calculation of components of vector crossproducts. To work out the **j** component, it helps to write the x components as an additional line to the right of the determinant.
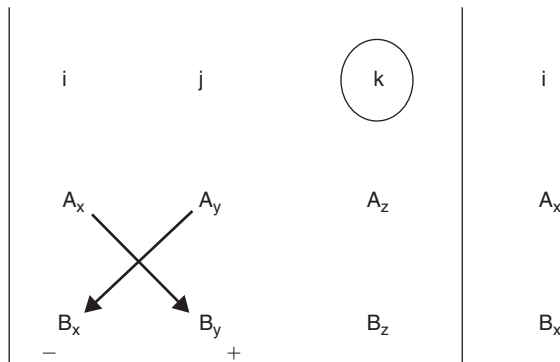
■ **EXAMPLE 2.9**

Torque can be expressed as a crossproduct. For example, take a thin wheel that is free to rotate about an axis through its center at point O. A force **F** acts at the edge of the wheel at a point whose position relative to the center of the wheel O is given by the position vector **r** as shown. The force **F** tends to rotate the wheel (assumed initially at rest) counter-clockwise, so the angular velocity **ω** will point out of the page toward the reader.
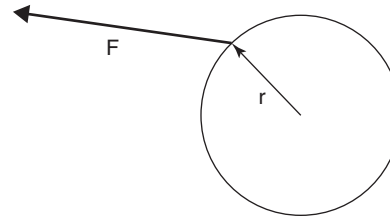
The torque caused by **F** starts the wheel rotating counterclockwise so that **ω** points out of the page. The **torque** vector is defined as $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$ and points out of the page. The magnitude of the torque is $\tau = r F \sin\theta$.

■ **EXAMPLE 2.10**

To work out the moment of a force about a point X, imagine a force **F** defined as:

$$\boldsymbol{F} = 1\boldsymbol{i} + 2.7\boldsymbol{j} + 3.5\boldsymbol{k} \text{ acting at a point } \boldsymbol{r} = 10\boldsymbol{i} + 0\boldsymbol{j} - 3.7\boldsymbol{k}$$

The moment of the force about the origin would be:

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} = (0 \times 3.5 - (-3.7) \times 2.7, -3.7 \times 1 - 10 \times 3.5, 10 \times 2.7 - 0 \times 1)$$
$$= (9.99, -38.7, 27)$$

This implies that the moment about the x axis is:

$$M_x = 9.99 \text{ Nm}$$
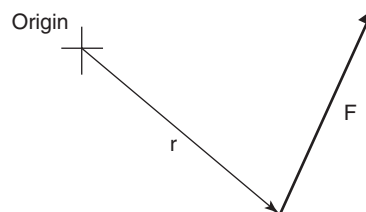
that about the y axis is:

$$M_y = -38.7 \text{ Nm}$$

and that about the z axis is:

$$M_z = 27 \text{ Nm}$$

2.4   System of Units

This book is written using the SI (système internationale) system of units. This takes the fundamental units of mass, length, and time as the kilogram, meter, and second, respectively. In this section, the units commonly used in motion analysis and me-

Table 2.4 Commonly used units in the systéme internationale (SI) system

Physical Quantity	SI Unit	Alternative Unit Name	Conversion Factor to Other Units
Mass	kg	Kilogram	1 kg = 2.2 lb
Length	m	meter	1 m = 39 in
Displacement	m	Meter	1 m = 3.281 ft
Time	s	Second	1 s = 0.0002777h
Speed	m/s	ms^{-1}	1 m/s = 2.237 mph
Velocity	m/s	ms^{-1}	1 m/s = 3.281 ft/s
Acceleration	m/s^2	ms^{-2}	1 "gee" = 9.81 m/s^2
Force	N	Newton	1 N = 0.1 kg weight
Moment	Nm	Newton-meter	1 Nm = 0.72 lb/ft
Impulse	Ns	Newton second	
Pressure	N/m^2	Pascal	1 N/m^2 = 0.001422 psi
Linear momentum	kg m/s		
Angle or angular displacement	rad	Radians	1 rad = 57.3°
Angular speed	rad/s		1 rad/s = 57.3 °/s
Angular acceleration	rad/s^2	rad/s^{-2}	
Torque	Nm		1 Nm = 0.72 lb/ft
Angular momentum	$kg/m^2/s$		
Angular impulse	Nm		
Power	J/s	Watt	1 J/s = 0.001343 hp
			1 kW = 1.341 hp

chanics are summarized. Table 2.4 shows the physical quantity, its SI unit and any alternative names. The table also shows the conversion of some of the SI units to different systems in common use. Although very little use is made of alternative units in this book, it is useful to have a point of reference in case they are encountered in everyday situations. For example, a baseball pitcher releases the ball at 100 mph. In this case, miles per hour are used for two reasons:

1. The unit of miles per hour is well known by most people, so they can relate to it easily.
2. The unit of miles per hour is a convenient size so that the numbers obtained are neither too big nor too small.

Also, in this book, units such as m/s and ms^{-1} (meters per second) are used interchangeably. More information on SI units is given in Appendix 1.

SUMMARY

This chapter has introduced the fundamental concepts of scalar and vector quantities. Addition, subtraction, and multiplication of scalars and vectors are considered using graphical methods, trigonometric methods, and unit vectors. Three case studies are used to illustrate practical applications of the use of vectors to human movement situations. Scalar products are used for calculating the amount of work done when the force and displacement vectors are at some arbitrary angle to each other. Vector products are used for calculating the moment caused by a force acting at some position r using both formula and determinant methods.
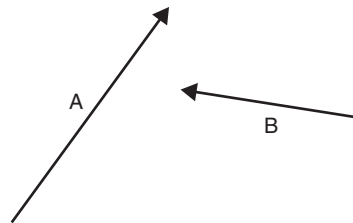
REFERENCE

Spiegel MR: *Schaum's Outline of Vector Analysis.* New York. Schaum Publishing; 1959.

STUDY QUESTIONS

1. Classify the following as vectors or scalars:

   Force, torque, acceleration, velocity, mass, volume, displacement, area, length, and angular velocity

2. Add the following vectors **A** and **B** together by the tip-to-tail approach and draw in the resultant vector **C**.

3. Use the polygon of vectors method to find the resultant of the following force vectors:

   20 N acting west

   35 N acting at 60° northeast

   25 N acting north

   15 N acting at 30° southwest

   Draw these on graph paper and do not forget to join them tip to tail. From the graph paper, measure the magnitude and direction of the resultant. Use a ruler and protractor for these measurements.

4. An ice hockey puck has a velocity of 15 m/s parallel to the edge of the rink. It also has a velocity of 5 m/s perpendicular to this edge. Use the Pythagorean theorem to calculate the magnitude of the resultant velocity of the puck to two decimal places.

5. Use the parallelogram rule to find the resultant of the following two vectors **A** and **B**.


   The vector **B** makes an angle of 55° with **A**. Draw these on graph paper, form the completed parallelogram, and measure the size *and* direction of the resultant. Repeat the calculation of the resultant, this time using the cosine rule. Compare your answers to deduce something about the errors involved in the first method.

6. The following vectors represent the wind velocity at a point adjacent to the long jump area at an Olympic Games. In (a), the vector corresponds to a wind velocity of 10 km/h and is a tail wind. Comment on the wind velocities for (b), (c), and (d). The wind velocity has to be below 5 km/h

for the long jump competition to proceed. Under which condition (or conditions), (a) to (d), would you allow the competition to go ahead?

(a)　　　　(b)　　　　(c)　　　　(d)
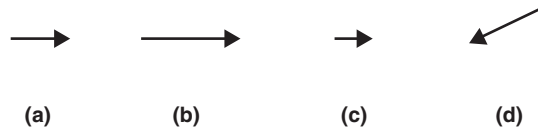
7. A golf ball is projected with a velocity of 45 m/s at an angle of 10° with respect to the horizontal. What is the horizontal component of this velocity?

8. A javelin is projected with an initial velocity of 40 m/s. The vertical component of this velocity is measured as 29 m/s. What is its initial horizontal velocity?

9. The high bar of a piece of gymnastic equipment is subject to two forces as a gymnast swings around the bar. One is a vertical up force supplied by the vertical support members at the sides of the high bar. This force is 750 N upward on the bar itself. The other force is a force on the bar acting down and to the right at an angle of 30° to the horizontal applied by the gymnast's hands as he swings. This force amounts to 510 N. Draw these forces on a force diagram showing the two applied forces and the direction of the resultant. Use the cosine law to work out the magnitude and direction of the resultant. (It is assumed in this question that the two vertical supports have identical forces in them. In this case, 375 N each. These two forces are assumed to be combined into one 750-N force.)

10. An acceleration vector has a magnitude of 16 ms^{-2} and is directed upward at an angle of 35° to the horizontal. What are the vertical and horizontal components of the acceleration along the x and y axes? The x and y axes are directed horizontally and vertically such that the acceleration vector is in the x–I plane.

11. Unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are set up along the x, y, and z axes.
    (a) A force is expressed as $\mathbf{F} = 12\,\mathbf{i} + 25\mathbf{j} + 0\mathbf{k}$. What are the components of the force along the x, y, and z axes?
    (b) A displacement vector is given by $\mathbf{d} = 2\,\mathbf{i} + 0\,\mathbf{j} + 3\,\mathbf{k}$. What is the total displacement in meters?
    (c) Two velocities given by $v_1 = 10\,\mathbf{i} + 5\,\mathbf{j} - 3\,\mathbf{k}$
    and $v_2 = -6\,\mathbf{i} + 3\,\mathbf{j} + 5\mathbf{k}$ are added vectorially to give a resultant.
    Write an expression for the resultant $v = v_1 + v_2$ using the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

12. A rower exerts a force on his oars of $\mathbf{F} = 200$ N acting horizontally while the displacement of the oars is 1.2 m at an angle of 15° above the horizontal.

d = 0.9 m
15°
F = 200 N
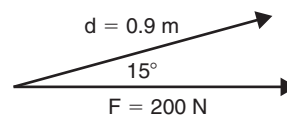
What is the work done as a result of this force and displacement?

13. A rower exerts a force on his oars given by $\mathbf{F} = 150\,\mathbf{i} + 45\,\mathbf{j} + 0\,\mathbf{k}$. The displacement of the oar handles is given by $\mathbf{d} = 0.8\,\mathbf{i} + 0.75\,\mathbf{j} + 0.15\,\mathbf{k}$. What is the work done?

14. A force of 50 N is applied tangentially to the outer rim of a discus (i.e., at right angles) to the line joining the center of the discus to the point on the rim. The diameter of the discus is 25 cm. What is the torque supplied by this force? Describe the effect of this torque on the discus.

50 N

12.5 cm
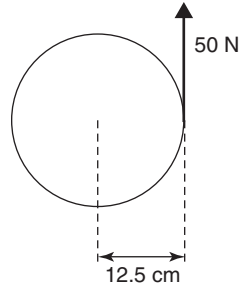
15. A force given by $F = 17\mathbf{i} + 25\,\mathbf{j} + 0\,\mathbf{k}$ is applied to a point A on the surface of a ball. The position vector of the point A relative to the center of the ball is given by $r = 0.05\,\mathbf{i} + 0.04\,\mathbf{j} - 0.02\,\mathbf{k}$. What is the torque supplied to the ball by this force?

16. Calculate the magnitude of the force in question 13. What is the diameter of the ball?

17. A soccer ball is moving with a velocity given by:

$$v = 10\mathbf{i} - 2.5\mathbf{j} + 3\mathbf{k}$$

What is the speed of the ball? Write down an expression for a unit vector pointing along the direction of v. If the drag force on the ball is given by $F_d = 0.015 \times (\text{speed})^2$, write down an expression for the drag force in terms of unit vectors i, j, and k.